Physics 323. Problem set IV. Thursday, May 1, 2003

DUE THURSDAY, MAY 8, 2003

Problem 1. (25 points)

The energy and the linear momentum of a distribution of electromagnetic fields in vacuum is given (in Gaussian units) by

$$U = \frac{1}{8\pi} \int d^3r \left(\vec{E}^2 + \vec{B}^2 \right) \tag{1}$$

$$\vec{P} = \frac{1}{4\pi c} \int d^3r \vec{E} \text{ x } \vec{B} \tag{2}$$

where the integration is over all space. Consider an expression for the electric and magnetic fields in terms of plane waves:

$$\vec{E}(\vec{r}, t) = \frac{1}{2} \sum_{\lambda} \int \frac{d^3k}{(2\pi)^{3/2}} \left[E_0(\vec{k}, \lambda) \hat{\epsilon}_\lambda(\vec{k}) e^{i(\vec{k}\vec{r} - \omega t)} + \ c.c. \right], \tag{3}$$

$$\vec{B}(\vec{r}, t) = \frac{1}{2} \sum_{\lambda} \int \frac{d^3k}{(2\pi)^{3/2}} \left[E_0(\vec{k}, \lambda) \hat{k} \text{ x } \hat{\epsilon}_\lambda(\vec{k}) e^{i(\vec{k}\vec{r} - \omega t)} + \ \text{c.c.} \right], \tag{4}$$

where $E_0(\vec{k}, \lambda)$ is a complex amplitude and c.c. stands for complex conjugate of the preceding term and $\omega = |\vec{k}|c$. The polarization satisfies $\hat{\epsilon}_\lambda(-\vec{k}) = \vec{\epsilon}^*_\lambda(\vec{k})$, and $\hat{\epsilon}_\lambda \hat{\epsilon}^*_{\lambda'} = \delta_{\lambda\lambda'}$.

(a) Show that $\vec{P}$ can be written as

$$\vec{P} = \frac{1}{8\pi c} \sum_{\lambda} \int d^3k |E_0(\vec{k}, \lambda)|^2 \hat{k} \tag{5}$$

Note that all time dependence has cancelled out. Explain.

(b) Obtain the corresponding expression for the total energy U. Employing the photon interpretation for each mode $(\vec{k}, \lambda)$ of the electromagnetic field, justify the statement that photons are massless.

Problem 2. (50 points)

Imagine a point particle of charge q moving with velocity v_x along the x-axis. At the points $x = \pm\, a$, $y = z = 0$, there are two neutral (zero charge) massive particles, with masses much larger than the point particle mass. The charged particle collides elastically against these massive particles and, in a first approximation, for $|v_x| \ll c$, reverses the direction of motion preserving the magnitude of its momentum. If one neglects the short acceleration periods in which the velocity of the particle changes direction,

a. Determine the expression of the Lienard-Wiechert potential at an arbitrary point along the y axis, $x = 0$, $z = 0$, at a given time t.

b. Compare the expressions with the ones obtained in the case of a free moving particle.

c. Use these results to determine the value of the electric field at an arbitrary point along the y axis at the time t.

d. What is the direction of the electric field ? What would be the value and direction of the magnetic field generated by the charged particle ?

e. What is the value of the electric field at a point along the x axis, $x = x_P$, when $|x_P| \gg |x_a|$. How does it compare with the case of a charged particle moving with constant velocity ?

f. Is it correct to neglect the acceleration periods ? What do you think it would happen if we do not do that ? What will be the dominant component of the electric and magnetic fields at large distances ?

Problem 3. (25 Points)

The energy of a relativistic particle moving in an external central field may be written as

$$\mathcal{E} = \sqrt{c^2\vec{p}^2 + m^2c^4} + V(r). \tag{6}$$

The angular momentum $\vec{L}$ of the particle is conserved and is equal to

$$|\vec{L}| = r\ p_T \tag{7}$$

where p_T is the magnitude of the momentum of the particle in the direction perpendicular to the radius. Therefore,

$$\mathcal{E} = \sqrt{c^2 p_r^2 + \frac{c^2 L^2}{r^2} + m^2 c^4} + V(r). \tag{8}$$

This should be compared with the non-relativistic expression

$$\mathcal{E} = \frac{p_r^2}{2m} + \frac{L^2}{2m\ r^2} + V(r). \tag{9}$$

a. Assume that $V(r) = -K\ r^{-\alpha}$, with $K, \alpha > 0$, and assume that the transverse momentum of the particle $p_T \neq 0$ $(L \neq 0)$. What are the values of α for which the particle cannot reach the origin $r = 0$?
Hint: This will happen if p_r should become negative to preserve energy. Compare this with the non-relativistic case.

b. Assume $\alpha = 1$ (Coulomb potential). What are the conditions for which the particle can escape to infinity ? When will the particle be able to reach the origin ?

c. If we do not ignore the fields generated by the particle, the particle will tend to radiate due to its accelerated motion. Describe what would happen to the particle if it is unable to escape to infinity.

d. Bonus Points : Determine the motion of a relativistic particle moving in a Coulomb potential.